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# ON THE RECURRENCE FORMULA OF THE EULER ZETA FUNCTIONS

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Abstract. In this paper, we find a new recurrence formula of the Euler zeta functions.

## 1. Introduction

The Euler zeta function is defined as $\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$ for $s \in \mathbb{C}$. This function is one of the main topic in number theory and famous function throughout all fields in mathematics as well. Historically, this function and related results have great influence on developing mathematical theories.
In this article, we consider recurrence formula of this function for even integer $s$. In [2], Lee-Ryoo found the following recurrence formula of $\zeta_E(2s)$ for $s \in \mathbb{N}$ using Fourier series.

Theorem 1.1. (Theorem 4 of [2]) *For* $s \geq 2 (\in \mathbb{N})$ *and* $\zeta_E(2) = \frac{\pi^2}{12}$,

$$\zeta_E(2s) = \frac{(-1)^s (2\pi)^{2s}}{{}_{2s}P_{2s-1}} \{ \frac{1}{2^{2s+1}} \frac{2^{2s+1} - 12s^2 + 3}{(2s-1)(2s+1)} - \sum_{k=1}^{s-1} (-1)^k \frac{1}{(2\pi)^{2k}} \zeta_E(2k) ({}_{2s}P_{2k-1} - {}_{2s-2}P_{2k-1}) \}.$$

The proof is very elementary. They first considered a function $f(x) = x^{2m}$ for $-2 < x < 2$ and found Fourier coefficients. Then $f(x)$ can be written as

$$f(x) = \frac{2^{2m}}{2m+1} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2},$$

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where $a_n = \sum_{k=1}^{m}(-1)^{k+1}{}_{2m}P_{2k-1}2^{2m-2k+1}\frac{2^{2k}}{n^{2k}\pi^{2k}}\cos n\pi.$

By substituting $x = 1$, they got

$$\sum_{k=1}^{s}(-1)^{k}{}_{2s}P_{2k-1}\frac{1}{2^{2k}\pi^{2k}}\zeta_E(2k) = \frac{2s+1-2^{2s}}{(2s+1)(2^{2s+1})}.$$

In the case of $m = 1$, one easily gets $\zeta_E(2) = \frac{\pi^2}{12}$. Using above equation, it is easy to get the formula in the theorem. See [2] for more details.

The goal of this article is to get more refined version of the recurrence formula of the Euler zeta functions.

## 2. Preliminaries

In this section, we briefly introduce Fourier series and Euler numbers. For more details, see [3] and [1].

For a real valued function $f(x)$ defined on $(-p, p)$, a trigonometric series $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos\frac{k\pi x}{p} + b_k \sin\frac{k\pi x}{p})$ is called the Fourier series of the function $f$ where $a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$, $a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx$, $b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx$. Then, this series converges to $f$ at the points of continuity.

The Euler number $E_n$ is defined by $\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}$, for $|t| < \pi$. Then it is known that $\zeta_E(2n) = \frac{(-1)^n\pi^{2n}(2-4^n)}{2(2n-1)!(1-4^n)}E_{2n-1}$ (See [1]).

## 3. The recurrence formula

The following is the main result.

Theorem 3.1. *For $s \geq 2(\in \mathbb{N})$ and $\zeta_E(2) = \frac{\pi^2}{12}$,*

$$\zeta_E(2s) = \frac{(-1)^s(\pi)^{2s}}{{}_{2s}P_{2s-1}}\Big\{\frac{1}{(2s-1)(2s+1)} - \sum_{k=1}^{s-1}(-1)^k\frac{1}{\pi^{2k}}\zeta_E(2k)({}_{2s}P_{2k-1} - {}_{2s-2}\,P_{2k-1})\Big\}.$$

*Proof.* We'll use the same function $f(x) = x^{2m}$ for $-2 < x < 2$. As in the proof of [2],

$$f(x) = \frac{2^{2m}}{2m+1} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2},$$

where $a_n = \sum_{k=1}^{m}(-1)^{k+1}{}_{2m}P_{2k-1}2^{2m-2k+1}\dfrac{2^{2k}}{n^{2k}\pi^{2k}}\cos n\pi.$
Now we substitute $x = 0$. Then,

$$\begin{aligned}
0 &= \frac{2^{2m}}{2m+1} + \sum_{n=1}^{\infty}\sum_{k=1}^{m}(-1)^{k+1}({}_{2m}P_{2k-1})2^{2m-2k+1}\frac{2^{2k}}{n^{2k}\pi^{2k}}\cos n\pi \\
&= \frac{2^{2m}}{2m+1} + \sum_{n=1}^{\infty}\sum_{k=1}^{m}(-1)^{k+1}({}_{2m}P_{2k-1})2^{2m+1}\frac{(-1)^n}{n^{2k}\pi^{2k}} \\
&= \frac{2^{2m}}{2m+1} + \sum_{k=1}^{m}(-1)^{k}({}_{2m}P_{2k-1})2^{2m+1}\frac{1}{\pi^{2k}}\zeta_E(2k).
\end{aligned}$$

Therefore $\sum_{k=1}^{s}(-1)^k{}_{2s}P_{2k-1}\dfrac{1}{\pi^{2k}}\zeta_E(2k) = -\dfrac{1}{2(2s+1)}$ and hence we also get $\sum_{k=1}^{s-1}(-1)^k{}_{2s-2}P_{2k-1}\dfrac{1}{\pi^{2k}}\zeta_E(2k) = -\dfrac{1}{2(2s-1)}$. By subtracting the second equation from the first equation, we get the result. □

Corollary 3.2. *For $s \geq 2(\in \mathbb{N})$ and $\zeta_E(2) = \frac{\pi^2}{12}$,*

$$\zeta_E(2s) = \frac{(-1)^s\pi^{2s}}{2s-1}\{\frac{1}{(2s+1)!} - \frac{1}{s}\sum_{k=1}^{s-1}(-1)^k\frac{1}{\pi^{2k}}\zeta_E(2k)\frac{(2k-1)(2s-k)}{(2s-2k+1)!}\}.$$

*Proof.* Since ${}_{2s}P_{2s-1} = (2s)!$ and
${}_{2s}P_{2k-1} - {}_{2s-2}\,P_{2k-1} = \dfrac{2(2s-2)!(2k-1)(2s-k)}{(2s-2k+1)!}$, the corollary immediately follows. □

As an application, we also get the following result as well.

Corollary 3.3. *For* $s \geq 2(\in \mathbb{N})$ *and* $\zeta_E(2) = \frac{\pi^2}{12}$,

$$\zeta_E(2s) = \frac{(-1)^s \pi^{2s}}{2s-1}\{\frac{1}{(2s+1)!} - \frac{1}{s}\sum_{k=1}^{s-1} \frac{(2s-k)(2-4^k)}{2(2k-2)!(1-4^k)(2s-2k+1)!}E_{2k-1}\}.$$

*Proof.* The proof is direct by substituting
$\zeta_E(2n) = \dfrac{(-1)^n \pi^{2n}(2-4^n)}{2(2n-1)!(1-4^n)}E_{2n-1}$ in the above corollary. $\square$

Remark 3.4. One can get many such recurrence formulas by considering the Fourier series of various functions. Even with the same function as in this paper, one can also get many forms of formulas by substituting other values into its Fourier series.

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